# NEHARI-TYPE FAMILIES OF HARMONIC MAPPINGS

HUGO ARBELÁEZ, MARTIN CHUAQUI, AND WILLY SIERRA

ABSTRACT. We introduce affine and linearly invariant families of locally injective harmonic mappings of the unit disk $\mathbb{D}$. We derive sharp distortion theorems for the Jacobian that are used to establish a uniform modulus of continuity for the quasiconformal mappings in each class. Finally, we find a converse of a recent theorem of Chen and Ponnusamy characterizing when the image $f(\mathbb{D})$ under a quasiconformal harmonic univalent mapping is a John domain.

## 1. INTRODUCTION

The purpose of this paper is to introduce certain Nehari-type classes $NH_\mu$ of locally injective harmonic mappings defined in the unit disk $\mathbb{D}$. In our study we will choose the parameter $\mu \in (0, 1]$. The classes are affine and linearly invariant, and are defined in terms of a Schwarzian derivative. It has been shown in [8] that for sufficiently small $\mu$, the classes consist of univalent mappings, but an explicit estimate is not known, let alone the sharp value of univalence. There is a rich literature on linear invariant families of holomorphic mappings since the original work of Pommerenke [9], [10], and also of families of this type of harmonic mappings [12]. The range chosen for the parameter $\mu$ is so that techniques from the Sturm theory and arguments based con convexity become applicable. The loss of conformality forces us to restrict the attention to mappings which are quasiconformal (in $\mathbb{D}$), but this is enough to derive an explicit uniform modulus of continuity depending on $\mu$, and thus, an extension to the closed disk of the quasiconformal mappings in the classes. The natural question of classifying *extremal mappings*, that is, univalent mappings that fail to be injective on the boundary, must confront the difficulty that the value of $\mu$ for univalency is not known, and will not be addressed here. We refer the reader to [6] for the holomorphic case.

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The first author was partially supported by the Universidad Nacional de Colombia, Hermes code 41381.

The third author wishes to thank the Universidad del Cauca for supporting this work through research project VRI ID 4671.

The authors were partially supported by Fondecyt Grant #1150115.

*Key words: Harmonic mappings, Schwarzian derivative, Nehari class, John domain.*

*2000 AMS Subject Classification.* Primary: 30C99, 30C65; Secondary: 31A05.

In [2], the authors study properties of John domains that are images of $\mathbb{D}$ under quasiconformal univalent harmonic mappings. They extend a classical characterization of such domains determining the rate of growth of the derivative of the Riemann mapping $f$ [6, Theorem 5.2], using the Jacobian instead. Furthermore, they show that a limes superior condition for the harmonic pre-Schwarzian at the boundary together with quasiconformality are sufficient for the image to be a John domain. Here no reference to a Nehari class is needed. As a harmonic analogue of Theorem 4 in [4], we establish the necessity of such a limes superior condition when $f \in NH_1$ is quasiconformal.

Over the last decade, a fair amount of classical results of geometric function theory dealing with the Schwarzian derivative have been extended for harmonic mappings. In this, two complementary definitions have appeared, one in [5], and later one by Hernández and Martín [7]. The results in this paper will be based on this second definition, which seems better suited when one is not to consider the Weierstarss-Enneper lift.

Let $f = h + \bar{g}$ be a sense-preserving harmonic mapping from $\mathbb{D}$ in $\mathbb{C}$. The Schwarzian derivative of $f$ is defined by

$$S_f = \partial_z P_f - \frac{1}{2}(P_f)^2,$$

where

$$P_f = \partial_z \log J_f = \frac{h''}{h'} - \frac{w'\bar{w}}{1-|w|^2}$$

is the pre-Schwarzian derivative, $J_f = |h'|^2 - |g'|^2$ the Jacobian and $w = g'/h'$ the second complex dilatation of $f$. If $f = u + iv$, the differential of $f$ is given by

$$D_f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

and $\|D_f\| := \sup\{|D_f z| : |z| = 1\} = |h'| + |g'|$ [7]. We observe that $\sqrt{J_f} \leq \|D_f\|$.

For $\mu \in (0,1]$ we denote by $NH_\mu$ the family of locally injective, sense-preserving harmonic mappings $f = h + \bar{g}$ for which

$$|S_f(z)| + \frac{|w'(z)|^2}{(1-|w(z)|^2)^2} \leq \frac{2\mu}{(1-|z|^2)^2}\,, \tag{1}$$

and we let $NH_\mu^0$ stand for subfamily of functions with $\nabla J_f(0,0) = (0,0)$.

It is not difficult to verify that the classes $NH_\mu$ are preserved under the changes $f \to af + b\bar{f}$, $a, b \in \mathbb{C}$, $|a| > |b|$, and the compositions $f \to f \circ \sigma$ for any automorphism $\sigma$ of the disk, and are therefore affine and linearly invariant.

Within these families it will be necessary to consider mappings which are *quasiconformal* in $\mathbb{D}$, as well as *univalent* mappings onto John domains. To be precise, we consider the class $NH_\mu(K) \subset NH_\mu$ of mappings for which

$$\|D_f\| \le K\ell(D_f) = K\inf\{|D_f(z)| : |z| = 1\} = K(|h'| - |g'|).$$

Thus, in $NH_\mu(K)$ we have

$$\sqrt{J_f} \le \|D_f\| \le \sqrt{K}\sqrt{J_f}\,. \tag{2}$$

The definition of a John domain will be postponed for the last section.

## 2. Preliminary Results

The first result in this section will be crucial throughout the paper.

**Lemma 1.** *Let $f \in NH^0_\mu$, $0 < \mu \le 1$.*
*(a) If $\mu = 1$ then*

$$|\partial_z \log J_f(z)| \le \frac{L''(|z|)}{L'(|z|)} = \frac{2|z|}{1 - |z|^2}\,, \tag{3}$$

*where*

$$L(z) = \frac{1}{2}\log\frac{1+z}{1-z}\,.$$

*If equality holds at a single $z \neq 0$ then $f$ is an affine mapping of $L$.*
*(b) If $0 \le \mu < 1$ then*

$$|\partial_z \log J_f(z)| \le \frac{A''_\mu(|z|)}{A'_\mu(|z|)} \le \frac{2\mu|z|}{1 - |z|^2}\,, \tag{4}$$

*where*

$$A_\mu(z) = \frac{1}{\beta}\frac{(1+z)^\beta - (1-z)^\beta}{(1+z)^\beta + (1-z)^\beta}\ \ ,\ \ \beta = \sqrt{1-\mu}\,.$$

*If equality holds at a single $z \neq 0$ then $f$ is an affine mapping of $A_\mu$.*

*Proof.* The proofs of parts (a) and (b) follow the same arguments. We will show that in $\mathbb{D}$

$$|\partial_z \log J_f(z)| \le v(|z|)\,, \tag{5}$$

where $v$ is the solution of the initial value problem

$$\begin{cases} v'(t) = \frac{1}{2}v^2(t) + \frac{2\mu}{(1-t^2)^2} \\ v\,(0) = 0\,. \end{cases} \tag{6}$$

Since the quantities involved in the estimates are invariant under rotations, it is sufficient to analyze the case when $z \in (0,1)$. For $y(t) = P_f(t), t \in [0,1)$ we have

$$y'(t) = \partial_z P_f(t) + \partial_{\bar z} P_f(t).$$

With $\varphi(t) = |y(t)|$ it follows that $\varphi'(t) \le |y'(t)| = |\partial_z y + \partial_{\bar z} y|$ , hence

$$\begin{aligned} \varphi'(t) &\le \left| S_f(t) + \frac{1}{2} y^2(t) - \frac{|w'(t)|^2}{(1-|w(t)|^2)^2} \right| \\ &\le \left| S_f(t) - \frac{|w'(t)|^2}{(1-|w(t)|^2)^2} \right| + \frac{1}{2}\varphi^2(t) \qquad (7) \\ &\le \frac{2\mu}{(1-t^2)^2} + \frac{1}{2}\varphi^2(t). \end{aligned}$$

Comparing (6) and (7) we have

$$\begin{cases} (\varphi - v)'(t) \le \frac{1}{2}(\varphi - v)(\varphi + v)(t), \\ (\varphi - v)(0) = 0 \end{cases}$$

and in consequence $|y(t)| = \varphi(t) \le v(t)$. The function $v(t)$ is given by

$$v(t) = \frac{F''(t)}{F'(t)}$$

for $F$ a function with $SF(t) = 2\mu(1-t^2)^{-2}$ and $F''(0) = 0$. For $\mu = 1$ we may take $F = L$, while for $\mu < 1$ we can choose $F = A_\mu$. This proves (2) and (3), except for the estimate on $A''_\mu / A'_\mu$, which can be found, for example, in [3]. Observe that the proof also shows that $(\varphi - v)'(t) \le 0$ for all $0 \le t < 1$.

Suppose now, without loss of generality, that there is $r \in (0,1)$ such that $\varphi(r) = v(r)$. Then, because of $\varphi - v \le 0$ and $(\varphi - v)'(t) \le 0$, one has $\varphi = v$ in $[0, r]$ and therefore

$$\varphi'(t) = \frac{2\mu}{(1-t^2)^2} + \frac{1}{2}\varphi^2(t)$$

for all $t \in [0, r]$. So, we must have equality in all the inequalities of (7) in $[0, r]$, from which it follows that

$$|S_f(t)| + \frac{|w'(t)|^2}{(1-|w(t)|^2)^2} \le \frac{2\mu}{(1-t^2)^2} = \left| S_f(t) - \frac{|w'(t)|^2}{(1-|w(t)|^2)^2} \right|,$$

for all $t \in [0, r]$. Hence $S_f \le 0$ in $[0, r]$ unless $w' \equiv 0$. Likewise, we conclude that $y^2(t) \le 0$ unless $w' \equiv 0$. Now, if $y^2(t) \le 0$ and $S_f(t) \le 0$, we see that

$$y(t) = \pm i|y(t)| = \pm i v(t) \qquad \text{and} \qquad \partial_z y \le 0,$$

and writing $y = l + is$, we get that in $[0, r]$, $l = 0$, $s = \pm v$,

$$2\partial_z y = (l_x + s_y) + (s_x - l_y)i \le 0 , \qquad \text{and} \qquad 2\partial_{\bar z} y = (l_x - s_y) + (s_x + l_y)i \le 0 .$$

It follows that $s_x = l_y$, $s_x = -l_y$, and $l_x = 0$, from where $s_y = \partial_z y \le 0$ and $\partial_{\bar z} y = -s_y$. As $\partial_{\bar z} y \le 0$, we obtain a contradiction unless $w' \equiv 0$, which implies that $f = F + \alpha \overline{F}$, for some $\alpha \in \mathbb{C}$ and $F$ an analytic function with $F''(0) = 0$ and $SF(z) = 2\mu(1-z^2)^{-2}$. This finishes the proof. $\square$

**Theorem 1.** *Let $f \in NH^0_\mu$, $0 < \mu \le 1$, such that $J_f(0) = 1$.*
*(a) If $\mu = 1$ then*

$$\frac{1}{L'(|z|)^2} \le J_f(z) \le L'(|z|)^2 \,.$$

*If equality holds at a point $z \neq 0$ then $f$ is an affine mapping of a rotation of $L$.*
*(b) If $\mu < 1$ then*

$$\frac{1}{A'_\mu(|z|)^2} \le J_f(z) \le A'_\mu(|z|)^2 \,.$$

*Equality holds at a single $z \neq 0$ then $f$ is an affine mapping of an analytic function.*

*Proof.* (a) Given $z \neq 0$, $z = re^{i\theta}$,

$$\log J_f(re^{i\theta}) = \int_0^r \frac{\partial}{\partial t} \log J_f(te^{i\theta}) d\theta = \int_0^r \left\langle \nabla \log J_f(te^{i\theta}), e^{i\theta} \right\rangle d\theta$$

and therefore, by (3),

$$\left|\log J_f(re^{i\theta})\right| \le 2 \int_0^r \left|\partial_z \log J_f(te^{i\theta})\right| dt \le 2 \int_0^r \frac{2t}{1-t^2} dt. \tag{8}$$

Hence follows the statement (a) of the theorem.

If there is equality in $z = re^{i\theta} \neq 0$, then

$$\int_0^r \left|\partial_z \log J_f(te^{i\theta})\right| dt = \int_0^r \frac{2t}{1-t^2} dt, \tag{9}$$

from where, by (3),

$$\left|\partial_z \log J_f(te^{i\theta})\right| = \frac{2t}{1-t^2}, \quad 0 \le t \le r,$$

which implies that $f$ is an affine mapping of a rotation of $L$.

To prove the statement (b) we follow the same idea as in the proof of (a). We use (4) to obtain

$$\left|\log J_f(re^{i\theta})\right| \le 2 \int_0^r \left|\partial_z \log J_f(te^{i\theta})\right| dt \le 2 \int_0^r \left\{ \frac{2t}{1-t^2} - \frac{2\mu^2}{1-t^2} A_\mu(t) \right\} dt,$$

where $\alpha = \sqrt{1-\mu}$. Now, proceeding as in the proof of Theorem 3 in [1], we see that

$$\left|\log J_f(re^{i\theta})^{1/2}\right| \le \log 4 \frac{(1-r)^{\mu-1}(1+r)^{\mu-1}}{((1+r)^\mu + (1-r)^\mu)^2} = \log A'_\mu(|z|),$$

from which we have

$$\frac{1}{A'_\mu(r)^2} \le J_f(re^{i\theta}) \le A'_\mu(r)^2.$$

Reasoning as in part (a), if there is equality for some $z \neq 0$ in any of the previous inequalities, then

$$\int_0^r \left|\partial_z \log J_f(te^{i\theta})\right| dt = \int_0^r v(t)dt,$$

where $v$ is defined as in equation (6). Hence, by (4),

$$\left|\partial_z \log J_f(te^{i\theta})\right| = v(t); \quad 0 \leq t \leq r,$$

which implies that $w$ is constant, and therefore $f = ag + b\overline{g}$, $a, b \in \mathbb{C}$ and $g$ analytic and univalent. □

The following corollaries can be established following the same arguments of the above proof, and we omit the details.

**Corollary 1.** *Suppose that $f \in NH_1^0$. For all $\xi \in \partial\mathbb{D}$ and all $0 \leq r < \rho < 1$,*

$$\left(\frac{1-\rho^2}{1-r^2}\right)^2 \leq \frac{J_f(\rho\xi)}{J_f(r\xi)} \leq \left(\frac{1-r^2}{1-\rho^2}\right)^2.$$

*In particular,*

$$\frac{1}{M_1^2} J_f(r\xi) \leq J_f(\rho\xi) \leq M_1^2 J_f(r\xi), \tag{10}$$

*if $0 \leq r < \rho < 1$ satisfies $1 - r^2 \leq M_1(1-\rho^2)$.*

**Corollary 2.** *Suppose that $f \in NH_1^0$ and let $z = re^{i\theta}$ and $w = re^{i\nu}$, $0 < r < 1$. Then*

$$e^{-2M_2} J_f(w) \leq J_f(z) \leq e^{2M_2} J_f(w),$$

*if $|\theta - \nu| \leq M_2(1-r)$.*

## 3. Boundary Behaviour

3.1. **Hölder continuity.** We will show that the functions in the family $NH_\mu(K)$ turn out to be bounded and Hölder continuous under a certain condition for the derivative of the pre-Schwarzian at zero.

**Theorem 2.** *Let $0 < \mu < 1$ and $f \in NH_\mu(K)$ such that $|y(0)| < 2\sqrt{1-\mu}$, where $y(z) = \partial_z \log J_f(z)$. Then*

*(a) $f$ is bounded. The condition $|y(0)| < 2\sqrt{1-\mu}$ is sharp.*

*(b) $f$ has a Hölder continuous extension to $\partial\mathbb{D}$.*

*Proof.* Given $0 \leq \theta < 2\pi$, we define the function

$$u(t) = u_\theta(t) = e^{-\frac{1}{2}\int_0^t \left|y(se^{i\theta})\right| ds}, \qquad 0 \leq t < 1.$$

Then $u$ satisfies the initial value problem

$$\begin{cases} u'' + qu = 0, \\ u(0) = 1, \\ u'(0) = -\frac{1}{2}\left|y(0)\right|, \end{cases}$$

where $q(t) = \frac{1}{2}|y(t)|' - \frac{1}{4}|y(t)|^2$. We note that $f \in NH_\mu(K)$ implies $q(t) \leq \dfrac{\mu}{(1-t^2)^2}$. Now, consider the initial value problem

$$\begin{cases} v'' + \frac{\mu}{(1-t^2)^2} v = 0, \\ v(0) = 1, \\ v'(0) = u'(0), \end{cases}$$

whose solution is

$$v(t) = \sqrt{1-t^2}\left\{C_1\left(\frac{1+t}{1-t}\right)^{\gamma} + C_2\left(\frac{1+t}{1-t}\right)^{-\gamma}\right\},$$

where

$$\gamma = \frac{\sqrt{1-\mu}}{2}, \qquad C_1 = \frac{1}{2} - \frac{|y(0)|}{8\gamma}, \quad \text{and} \quad C_2 = \frac{1}{2} + \frac{|y(0)|}{8\gamma}.$$

As $|y(0)| < 2\sqrt{1-\mu}$ then $C_1 > 0$ and so $v(t) > 0$ in $[0,1)$ and $v(1) = 0$. A standard comparison theorem guarantees that $u(t) \geq v(t)$ for all $t \in [0,1)$. Now, since $C_2 = 1 - C_1$ and $C_1 > 0$ it follows that

$$v^{-2}(t) \leq \frac{1}{C_1^2}\frac{(1+t)^{2\gamma-1}(1-t)^{2\gamma+1}}{[(1+t)^{2\gamma} - (1-t)^{2\gamma}]^2}.$$

From here and $u(t) \geq v(t)$ in $[0,1)$ we obtain

$$u^{-2}(t) \leq M\frac{1}{(1-t)^{1-2\gamma}}, \tag{11}$$

for all $0 < a \leq t < 1$ and some constant $M = M(C_1, a, \gamma) > 0$.

On the other hand, given $r \in (0,1)$ and $\theta \in [0, 2\pi)$

$$\begin{aligned} \left|\log\left(\frac{J_f(re^{i\theta})}{J_f(0)}\right)^{1/2}\right| &\leq \int_0^r \left|\nabla \log(J_f(te^{i\theta}))^{1/2}\right| dt \\ &= \int_0^r \left|\partial_z \log(J_f(te^{i\theta}))\right| dt \\ &= \log u^{-2}(r). \end{aligned}$$

So, with this and (11) it follows that

$$\sqrt{J_f(re^{i\theta})} \leq M\sqrt{J_f(0)}\,\frac{1}{(1-r)^{1-2\gamma}}, \tag{12}$$

for all $r \in [a, 1)$.

(a) We first show that $f$ is bounded. Obviously $f$ is bounded in $\overline{D(0,a)}$. On the other hand, for $a<|z|<1$ and $z=re^{i\theta}$, by (2)

$$\begin{aligned}|f(z)| &\le |f(z)-f(ae^{i\theta})|+|f(ae^{i\theta})| \\ &\le \int_a^r \left\|D_f(te^{i\theta})\right\| dt + C \\ &\le \sqrt{K}\int_a^r \sqrt{J_f(te^{i\theta})}\, dt + C,\end{aligned}$$

whence by (12) one sees that

$$|f(z)| \le M\sqrt{K\, J_f(0)}\int_a^1 \frac{1}{(1-t)^{1-2\gamma}}dt + C = \widetilde{M}(1-a)^{2\gamma}+C.$$

To show that the condition $|y(0)|<2\sqrt{1-\mu}$ is optimal for $f\in NH_\mu(K)$, we consider the function

$$f(z)=\frac{A_\mu(z)}{1-\sqrt{1-\mu}A_\mu(z)},$$

where $A_\mu$ is defined as in Lemma 1. A straightforward calculation shows that $f\in NH_\mu(K)$ and $f$ is not bounded.

(b) Let $0<a<\rho<1$. There is $\delta>0$ such that for all $z_1, z_2$, with $\rho<|z_1|,|z_2|<1$ and $|z_1-z_2|<\delta$, the hyperbolic segment $\Gamma$ joining $z_1$ and $z_2$ satisfies $\Gamma\subset\{z\,|\,\rho<|z|<1\}=: A_\rho$. Now, by (2), (12) and an argument of Gehring and Pommerenke in [6], we have

$$\begin{aligned}|f(z_1)-f(z_2)| &\le \int_\Gamma \|D_f(\zeta)\|\,|d\zeta| \\ &\le M\sqrt{K}\sqrt{J_f(0)}\int_\Gamma \frac{|dz|}{(1-|\zeta|)^{1-2\gamma}} \\ &\le \frac{B}{\sqrt{1-\mu}}|z_1-z_2|^{\sqrt{1-\mu}},\end{aligned}$$

where $B$ is a constant that only depends on $K$. It follows that $f$ is Hölder continuous in $A_\rho$ and therefore $f$ has a Hölder continuous extension to $\overline{A_\rho}$. □

3.2. **Logarithmic continuity.** We will prove that every function $f=h+\bar{g}$ in $NH_1^0$ can be extended continuously to $\overline{\mathbb{D}}$. There is no loss of generality in assuming that $h(0)=0$ and $h'(0)=1$.

As in the proof of Lemma 1, if $y = P_f = \partial_z \log J_f$, we get that for any fixed $0 \leq \theta < 2\pi$, $\varphi(t) = |y(te^{i\theta})|$, $0 \leq t < 1$, satisfies

$$\begin{aligned} \varphi' \leq |y'| &= \left| S_f + \frac{1}{2}y^2 - \frac{|w'|^2}{(1-|w|^2)^2} \right| \\ &\leq \left| S_f - \frac{|w'|^2}{(1-|w|^2)^2} \right| + \frac{1}{2}\varphi^2 \\ &\leq \frac{2}{(1-t^2)^2} + \frac{1}{2}\varphi^2. \end{aligned} \tag{13}$$

Likewise, we can see that the function

$$u(t) = u_\theta(t) = e^{-\frac{1}{2}\int_0^t \varphi(se^{i\theta})ds}$$

satisfies

$$u'' + qu = 0, \qquad \text{with} \qquad q(t) \leq \frac{1}{(1-t^2)^2} := p(t).$$

We consider now the test function

$$u_f(z) = \frac{u_\theta(t)}{\sqrt{1-t^2}}; \qquad z = te^{i\theta},$$

and we show that it is hyperbolically convex along any ray from the origin, which means that for all $0 \leq \theta < 2\pi$, $\phi(s) = u_f(\gamma(s)e^{i\theta})$ satisfies $\phi'' \geq 0$, where $\gamma(s) = \frac{e^{2s}-1}{e^{2s}+1}$, $0 \leq s < \infty$, is the parametrization of $[0,1)$ by hyperbolic arc length. We first note that if $v(t) = \sqrt{1-t^2}$, then

$$v'(t) = -\frac{t}{\sqrt{1-t^2}} \qquad \text{and} \qquad v'' + \frac{1}{(1-t^2)^2}v = v'' + pv = 0.$$

Moreover, $\gamma' = 1 - \gamma^2 = (v \circ \gamma)^2$. Thus,

$$\phi' = \frac{vu' - uv'}{v^2}\gamma' = vu' - uv'$$

and therefore

$$\phi'' = (u''v - v''u)\gamma' = (p-q)uv\gamma' \geq 0.$$

From this and the normalization $\nabla J_f(0,0) = (0,0)$, which implies that $\phi$ has a minimum at 0, we may conclude that, either $u_f$ is constant along some segment $[0, re^{i\mu})$, $0 < r < 1$, or $\phi(s) = u_f(\gamma(s)e^{i\theta})$ is strictly convex (hence strictly increasing) for all $0 \leq \theta < 2\pi$. We study these two cases separately.

**Case 1.** In the first case, without loss of generality, we can assume that $u_f$ is constant in $[0,1)$. Then, because of $u_f(0) = 1$, we have $u_f \equiv 1$ in $[0,1)$, which implies $u(t) = \sqrt{1-t^2}$, $0 \leq t < 1$, and consequently

$$\varphi(t) = \frac{2t}{1-t^2}, \qquad 0 \leq t < 1.$$

Since $\varphi$ satisfies

$$\varphi'(t) = \frac{2}{(1-t^2)^2} + \frac{1}{2}\varphi^2,$$

then we have equality in all the inequalities of (13). Therefore, as in the proof of Lemma 1, we conclude that $f \in NH_1^0$ has the form $f = h + \beta\overline{h}$, for some $\beta \in \mathbb{C}$, where $h$ is a rotation of

$$L(z) = \frac{1}{2}\log\frac{1+z}{1-z}, \qquad z \in \mathbb{D},$$

hence that $h$ (and therefore $f$) has a spherically continuous extension to $\overline{\mathbb{D}}$.

**Case 2.** Now suppose that $\phi(s) := \phi_\theta(s) = u_f(\gamma(s)e^{i\theta})$ is strictly convex for all $0 \le \theta < 2\pi$. We will use a standard argument to obtain a bound for $J_f$, which gives us the desired continuous extension of $f$ to $\overline{\mathbb{D}}$. Indeed, the proof will show that $f$ has a logarithmic modulus of continuity in $\overline{\mathbb{D}}$. The condition $\nabla J_f(0,0) = (0,0)$ implies that $\phi_\theta(s)$ is strictly increasing for all $\theta$. Therefore $\phi'_\theta(1) > 0$ for all $\theta$ and so, by continuity, there is $\delta > 0$ such that

$$\phi'_\theta(s) \ge \delta, \qquad 0 \le \theta < 2\pi \quad \text{and} \quad s \ge 1.$$

It follows that

$$\frac{u_\theta(\gamma(s))}{v(\gamma(s))} \ge \phi_\theta(1) + \delta(s-1), \qquad 0 \le \theta < 2\pi \quad \text{and} \quad s \ge 1,$$

and consequently

$$\frac{1}{u_\theta(t)} \le \frac{1}{\sqrt{1-t^2}}\frac{1}{\delta}\left(\frac{1}{2}\log\frac{1+t}{1-t} - 1\right)^{-1}, \qquad t \ge \frac{e-1}{e+1}.$$

Thus, for all $z = re^{i\theta} \in \mathbb{D}$,

$$e^{\int_0^r \left|\partial_z \log J_f(te^{i\theta})\right| dt} \le \frac{1}{\delta^2}\frac{1}{1-r^2}\left(\frac{1}{2}\log\frac{1+r}{1-r} - 1\right)^{-2}. \tag{14}$$

On the other hand,

$$\log\frac{J_f(re^{i\theta})}{J_f(0)} = \int_0^r \frac{\partial}{\partial t}\log J_f(te^{i\theta})dt,$$

from where

$$\left|\log\frac{J_f(re^{i\theta})}{J_f(0)}\right| \le 2\int_0^r \left|\partial_z \log J_f(te^{i\theta})\right| dt.$$

Thus, by (14),

$$\sqrt{J_f(re^{i\theta})} \le \sqrt{J_f(0)}\, e^{\int_0^r \left|\partial_z \log J_f(te^{i\theta})\right| dt} \le C\frac{1}{1-r^2}\left(\frac{1}{2}\log\frac{1+r}{1-r} - 1\right)^{-2}$$

and therefore, by (2),

$$\|D_f(z)\| \le \frac{M}{1-r^2}\left(\frac{1}{2}\log\frac{1+r}{1-r} - 1\right)^{-2},$$

for some constant $M$ independent on $z$. We may now conclude, by integration along hyperbolic geodesics in $\mathbb{D}$, see for example proof of Theorem 2 in [6], that for all $z, z' \in \mathbb{D}$,

$$\left|f(z) - f(z')\right| \leq M_1 \left(\log \frac{M_2}{|z - z'|}\right)^{-1},$$

where $M_1, M_2 > 0$ are constants independent on $z$ and $z'$, which is the desired conclusion.

## 4. John Domains

Finally, in this section we establish a partial converse to Theorem 5 in [2], and a harmonic analogue of part (ii) of Theorem 4 in [4]. We recall the definition of a John domain in the plane.

**Definition 1.** *Let $b > 1$. A domain $\Omega \subset \mathbb{C}$ is said to be a $b-$domain of John, if there exist $p \in \Omega$ such that every point $q \in \Omega$ can be joined to $p$ by a rectifiable curve $\gamma \subset \Omega$ with*

$$\ell(\gamma(y, q)) \leq b\, d(y, \partial\Omega) \text{ for all } y \in \gamma,$$

*where $\gamma(y, q)$ is the subarc of $\gamma$ from $y$ to $q$, $\ell(\gamma(y, q))$ its length, and $d(y, \partial\Omega)$ the distance from $y$ to the boundary of $\Omega$.*

The point $p$ in this definition will be referred to as the center of the John domain. If $f : \mathbb{D} \to \mathbb{C}$ is a univalent mapping, we will say that $\Omega = f(\mathbb{D})$ is a *radial John domain*, if $p = f(0)$ and $\gamma$ can be chosen always to be the image of some radial segment $[0, z]$.

We begin with a variant for the class $NH_1^0(K)$ of a theorem proved in [11] for conformal mappings. In [13] an analogue of this theorem is established for the lift of a harmonic mapping.

**Lemma 2.** *Suppose that $f \in NH_1^0(K)$ and $\Omega = f(\mathbb{D})$ is a radial John domain. Then there are constants $M = M(K) > 0$ and $\delta = \delta(K) \in (0, 1)$ such that*

$$\|D_f(\rho\xi)\| \leq M\, \|D_f(r\xi)\| \left(\frac{1 - \rho}{1 - r}\right)^{\delta - 1}$$

*for all $\xi \in \partial\mathbb{D}$ and $0 \leq r < \rho < 1$.*

*Proof.* Let $z \in \mathbb{D}$. Proceeding as in the proof of Theorem 1 in [2] we obtain

$$\|D_f(z)\| \geq \frac{1 + K}{K} \frac{d_f(z)}{1 - |z|^2}, \tag{15}$$

here $d_f(z)$ is the Euclidean distance from $f(z)$ to the boundary of $\Omega$. On the other hand, as $\Omega$ is a radial John domain (with center $f(0)$), there is $c > 0$ such that

$$\ell(f[r\xi, \rho\xi]) \leq c d_f(r\xi)$$

for all $\xi \in \partial\mathbb{D}$ and $0 \le r < \rho < 1$. From here

$$\begin{aligned}\frac{1}{K}\int_r^1 \|D_f(t\xi)\|\,dt &\le \int_r^1 \ell(D_f(t\xi))dt \\ &\le \int_r^1 |df(t\xi)|dt \\ &\le c d_f(r\xi).\end{aligned}$$

By (15) it follows that

$$\int_r^1 \|D_f(t\xi)\|\,dt \le M_3(1-r^2)\,\|D_f(r\xi)\|\,, \tag{16}$$

where $M_3 := \frac{cK^2}{1+K}$.

Now, for $\xi \in \partial\mathbb{D}$ fixed, we consider the function

$$\varphi(r) = \int_r^1 \|D_f(t\xi)\|\,dt,$$

which, by (16), satisfies

$$\varphi'(r) = -\|D_f(r\xi)\| \quad \text{and} \quad \varphi(r) \le M_3(1-r^2)\,\|D_f(r\xi)\|\,.$$

It follows that for $0 < r < \rho < 1$,

$$\log\frac{\varphi(\rho)}{\varphi(r)} = \int_r^\rho \frac{\varphi'(t)}{\varphi(t)}dt \le -\frac{1}{M_3}\int_r^\rho \frac{dt}{1-t^2} \le -\frac{1}{2M_3}\int_r^\rho \frac{dt}{1-t}$$

and therefore

$$\varphi(\rho) \le \varphi(r)\left(\frac{1-\rho}{1-r}\right)^{\frac{1}{2M_3}} \le M_3(1-r^2)\,\|D_f(r\xi)\|\left(\frac{1-\rho}{1-r}\right)^{\frac{1}{2M_3}} \tag{17}$$

for all $0 < r < \rho < 1$.

On the other hand, by (2), for all $0 < \rho < 1$,

$$\varphi(\rho) \ge \int_\rho^{\frac{1+\rho}{2}} \|D_f(t\xi)\|\,dt \ge \int_\rho^{\frac{1+\rho}{2}} \sqrt{J_f(t\xi)}\,dt$$

and since $\rho \le t \le \frac{1+\rho}{2}$ implies $\frac{1-\rho^2}{1-t^2} \le 2$, we obtain from (10) that

$$\varphi(\rho) \ge \frac{1}{2}\int_\rho^{\frac{1+\rho}{2}} \sqrt{J_f(\rho\xi)}\,dt = \frac{1-\rho}{4}\sqrt{J_f(\rho\xi)}\,.$$

From here and (2) we have

$$\varphi(\rho) \ge \frac{1}{8\sqrt{K}}(1-\rho^2)\,\|D_f(\rho\xi)\|\,.$$

It follows by (17) that

$$\frac{1}{8\sqrt{K}}(1-\rho^2)\,\|D_f(\rho\xi)\| \le M_3(1-r^2)\,\|D_f(r\xi)\|\left(\frac{1-\rho}{1-r}\right)^{\frac{1}{2M_3}}$$

and therefore

$$\frac{(1-\rho)\,\|D_f(\rho\xi)\|}{(1-r)\,\|D_f(r\xi)\|} \leq M\left(\frac{1-\rho}{1-r}\right)^{\delta},$$

where $M = 8M_3\sqrt{K}$ and $\delta = \frac{1}{2M_3}$. From where the lemma follows. □

**Theorem 3.** *If $f \in NH_1^0(K)$ and $f(\mathbb{D})$ is a radial John domain, then*

$$\limsup_{|z|\to 1}(1-|z|^2)\mathrm{Re}\,\{zP_f(z)\} < 2\,.$$

*Proof.* We define the function

$$\varphi(z) = P_f(z) = \partial_z \log J_f(z).$$

Since $f \in NH_1^0$, it follows from (3) that for $z \in \mathbb{D}$, $|z| = r$,

$$\begin{aligned}|\partial_z\varphi(z)| &= |S_f(z) + \frac{1}{2}\varphi^2(z)| \\ &\leq \frac{2}{(1-r^2)^2} + \frac{2r^2}{(1-r^2)^2} \\ &= \frac{d}{dr}\frac{2r}{1-r^2}.\end{aligned}$$

Similarly, since

$$\partial_{\bar z}\varphi(z) = \frac{\partial}{\partial z}\frac{w'(z)\bar w(z)}{1-|w(z)|^2} = -\left(\frac{|w'(z)|}{1-|w(z)|^2}\right)^2,$$

then

$$\begin{aligned}|\partial_z\varphi(z) + \partial_{\bar z}\varphi(z)| &= \left|S_f(z) - \left(\frac{|w'(z)|}{1-|w(z)|^2}\right)^2 + \frac{1}{2}\varphi^2(z)\right| \\ &\leq \frac{2}{(1-r^2)^2} + \frac{2r^2}{(1-r^2)^2} \\ &= \frac{d}{dr}\frac{2r}{1-r^2}.\end{aligned} \tag{18}$$

Arguing by contradiction let us assume that

$$\limsup_{|z|\to 1}(1-|z|^2)\mathrm{Re}\,\{zP_f(z)\} = 2.$$

Then there is a sequence $(z_n) \in \mathbb{D}$ such that $|z_n| \to 1$ and

$$\lim_{n\to\infty}(1-|z_n|^2)\mathrm{Re}\,\{z_nP_f(z_n)\} = 2. \tag{19}$$

Let us fix $x \in (0,1)$ and let

$$z_n = \rho_n\xi_n, \quad |\xi_n| = 1, \qquad \text{and} \qquad r_n = \sigma_n(x),$$

where $\sigma_n$ is the automorphism of $\mathbb{D}$ defined by

$$\sigma_n(z) = \frac{\rho_n - z}{1-\rho_n z}.$$

Note that $\lambda(r_n,\rho_n)=\lambda(x,0)$ for all $n$, where $\lambda$ is the hyperbolic metric in $\mathbb{D}$. It follows by (18) that for $0<r<\rho_n$,

$$\begin{aligned}
|\mathrm{Re}\,\{\xi_n\varphi(\rho_n\xi_n)\}-\mathrm{Re}\,\{\xi_n\varphi(r\xi_n)\}\,| &= \left|\int_r^{\rho_n}\frac{\partial}{\partial t}\mathrm{Re}\,\{\xi_n\varphi(t\xi_n)\}\,dt\right| \\
&= \left|\int_r^{\rho_n}\mathrm{Re}\,\{\xi_n^2(\partial_z\varphi(t\xi_n)+\partial_{\bar z}\varphi(t\xi_n))\}\,dt\right| \\
&\le \int_r^{\rho_n}|\partial_z\varphi(t\xi_n)+\partial_{\bar z}\varphi(t\xi_n)|\,dt \\
&\le \int_r^{\rho_n}\frac{\partial}{\partial t}\frac{2t}{1-t^2}dt \\
&= \frac{2\rho_n}{1-\rho_n^2}-\frac{2r}{1-r^2}.
\end{aligned}$$

Thus, if $0<r_n\le r\le\rho_n$, we have on the one hand

$$2-\frac{1-r^2}{r}\mathrm{Re}\,\{\xi_n\varphi(r\xi_n)\}\le\frac{\rho_n}{r}\frac{1-r^2}{1-\rho_n^2}\left[2-\frac{1-\rho_n^2}{\rho_n}\mathrm{Re}\,\{\xi_n\varphi(\rho_n\xi_n)\}\right]$$

and by other hand,

$$\frac{1-r^2}{1-\rho_n^2}\le\frac{1-r}{1-\rho_n}\le\frac{1-r_n}{1-\rho_n}=\frac{1+x}{1-x\rho_n}\le\frac{1+x}{1-x}.$$

Therefore, if $0<r_n\le r\le\rho_n$,

$$\left|2-\frac{1-r^2}{r}\mathrm{Re}\,\{\xi_n\varphi(r\xi_n)\}\right|\le\frac{\rho_n}{r_n}\frac{1+x}{1-x}\left|2-\frac{1-\rho_n^2}{\rho_n}\mathrm{Re}\,\{\xi_n\varphi(\rho_n\xi_n)\}\right|.$$

As $\rho_n=|z_n|\to1$ and $\lambda(r_n,\rho_n)=\lambda(x,0)$, then $r_n\to1$. Also, by (19), given $\epsilon>0$ there is $N=N(\epsilon,x)$ such that

$$\left|2-\frac{1-r^2}{r}\mathrm{Re}\,\{\xi_n\varphi(r\xi_n)\}\right|<\epsilon$$

for all $n\ge N$ and $r_n\le r\le\rho_n$. From here and the following equality

$$\begin{aligned}
\log\frac{(1-\rho_n^2)\sqrt{J_f(\rho_n\xi_n)}}{(1-r_n^2)\sqrt{J_f(r_n\xi_n)}} &= \int_{r_n}^{\rho_n}\frac{\partial}{\partial r}\log\left[(1-r^2)\sqrt{J_f(r\xi_n)}\right]dr \\
&= \int_{r_n}^{\rho_n}\left[-\frac{2r}{1-r^2}+\mathrm{Re}\,\{\xi_n\partial_z\log J_f(r\xi_n)\}\right]dr \\
&= \int_{r_n}^{\rho_n}\left[-\frac{2r}{1-r^2}+\mathrm{Re}\,\{\xi_n\varphi(r\xi_n)\}\right]dr,
\end{aligned}$$

we obtain

$$\log\frac{(1-\rho_n^2)\sqrt{J_f(\rho_n\xi_n)}}{(1-r_n^2)\sqrt{J_f(r_n\xi_n)}}>-\epsilon\int_{r_n}^{\rho_n}\frac{r}{1-r^2}dr=\log\left(\frac{1-\rho_n^2}{1-r_n^2}\right)^{\frac{\epsilon}{2}}.$$

As a consequence

$$\frac{(1-\rho_n^2)\sqrt{J_f(\rho_n\xi_n)}}{(1-r_n^2)\sqrt{J_f(r_n\xi_n)}} > \left(\frac{1-\rho_n^2}{1-r_n^2}\right)^{\frac{\epsilon}{2}}$$

for all $n \geq N$. Thus,

$$\frac{(1-\rho_n)\sqrt{J_f(\rho_n\xi_n)}}{(1-r_n)\sqrt{J_f(r_n\xi_n)}} > \frac{1}{2}\left(\frac{1-\rho_n}{1-r_n}\right)^{\frac{\epsilon}{2}}$$

for all $n \geq N$. It follows from (2) that

$$\frac{(1-\rho_n)\,\|D_f(\rho_n\xi_n)\|}{(1-r_n)\,\|D_f(r_n\xi_n)\|} > \frac{1}{2\sqrt{K}}\left(\frac{1-\rho_n}{1-r_n}\right)^{\frac{\epsilon}{2}} > \frac{1}{2\sqrt{K}}\left(\frac{1-x}{1+x}\right)^{\frac{\epsilon}{2}}$$

for all $n \geq N$. We conclude that for all $\beta > 0$ and all $x \in (0,1)$ there are points $\xi \in \partial\mathbb{D}$, $\rho \in (0,1)$ and $r = \frac{\rho-x}{1-\rho x}$ such that

$$\frac{(1-\rho)\,\|D_f(\rho\xi)\|}{(1-r)\,\|D_f(r\xi)\|} > \beta.$$

This leads us to a contradiction, since by Lemma 2, for all $\xi \in \partial\mathbb{D}$,

$$\frac{(1-\rho)\,\|D_f(\rho\xi)\|}{(1-r)\,\|D_f(r\xi)\|} \leq M\left(\frac{1-\rho}{1-r}\right)^{\delta} \leq M\left(\frac{1-x}{1+x}\right)^{\delta}$$

if $0 < r \leq \rho < 1$ satisfies $r = \frac{\rho-x}{1-\rho x}$. □

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Hugo Arbeláez, Escuela de Matemáticas, Universidad Nacional de Colombia, Medellín-Colombia

Martín Chuaqui, Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Santiago-Chile

Willy Sierra, Departamento de Matemáticas, Universidad del Cauca, Popayán-Colombia